This document contains
blank pages that were
not filmed

NWL TECHNICAL REPORT TR2446
August 1970

ACOUSTIC RAYS IN AN ARBITRARY MOVING INHOMOGENEOUS MEDIUM

by

Peter Uginčius and Bernd Zondek
Warfare Analysis Department

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

CONTENTS

FOREWORD

A better understanding of sound propagation in a complex medium like the ocean or atmosphere continues to be a goal of vital interest to the Navy. In many practical applications it is sufficient to approximate the ocean as a stationary medium in which the acoustic properties vary only with depth. The theory then is very simple. In a more realistic model, however, this theory is imbued with a high degree of complexity due to general three-dimensional inhomogeneities as well as motion of the medium.

This report addresses itself to the exact theory of ray acoustics in such a general medium. Differential equations for the acoustic rays are derived in two different forms, which are suitable for implementation on a digital computer.

This work was done in the Science Research Group of the Warfare Analysis Department under Independent Research Project Number R360FR103/2101/R01101001. The date of completion was 1 June 1970.

RELEASED BY:

David R. Brown Jr.

4 August 1970

DAVID R. BROWN, JR., Acting Head
Warfare Analysis Department

Abstract

Two different formulations are presented in terms of ordinary second-order vector differential equations for acoustic rays in a three-dimensional medium moving with an arbitrary velocity (either subsonic or supersonic), and having an arbitrary index of refraction. The first formulation has the arc length of the ray as independent variable, while the second one is given in terms of a canonical variable and is equivalent to Keller's Hamiltonian formulation [J. B. Keller, J. Appl. Phys. 25, 938-947 (1954)]. An explicit Lagrangian is constructed for this problem from which a generalized Fermat's principle is derived. An error in an earlier publication [P. Uginčius, J. Acoust. Soc. Amer. 37, 476-479 (1965)] is corrected, and it is shown that this error disappears when the velocity of the medium is small compared to the speed of sound. In that case, and with the further restriction of planar rays, an explicit expression is derived for the curvature of the rays in terms of the velocity, speed of sound, and their gradients.

INTRODUCTION

In Ref. 1 one of the authors derived a vector differential equation for the acoustic ray paths in a moving inhomogeneous medium in which the velocity field and index of refraction could be arbitrary functions of position. Unfortunately an error was made in that derivation so that the main result (Eq. 14, Ref. 1) is valid only for those cases in which the velocity of the medium is much smaller than the speed of sound. In this paper we derive the correct equation for arbitrary velocities, both supersonic and subsonic. This is done in two ways: in part I by the method of Ref. 1, and in Part II by finding the characteristics of the time-independent eikonal equation. The second method of derivation is important for two reasons: 1. being based on the rigorous theory of partial differential equations it demonstrates the validity of the "tricky" method used in Ref. 1; and, more important, 2. it leads to a natural (canonical) independent variable τ (different from arc length or time) in terms of which the differential equation for the ray paths assumes a simpler form. The latter method is quite analogous and equivalent to the Hamiltonian formalism used by Keller[2]. Keller's work, however, does not finish the problem, because it does not consider the second set of the canonical equations for the generalized momenta.

In part III we construct an explicit Lagrangian for this problem and derive a generalized Fermat's principle. The correct expression of Fermat's principle for a moving inhomogeneous medium had evidently not been known until now[3].

In part IV the general equations are specialized to planar rays in a medium with small velocity, and it is shown that the results of Ref. 1 are correct in this case.

I. DERIVATION OF THE DIFFERENTIAL EQUATION FOR THE RAYS

Let the acoustical properties of the medium be described by the speed of sound $c(\mathbf{r})$, and let its velocity be $\mathbf{V}(\mathbf{r})$, where both c and $\mathbf{V}$ are arbitrary well-behaved functions of the position vector $\mathbf{r}$. We shall use the following terminology:

$\mathbf{r}' = d\mathbf{r}/ds$ = unit tangent to a ray.

$n = c_0/c$ = index of refraction.

$\mathbf{v} = \mathbf{V}/c$ = nondimensional velocity.

$\hat{\mathbf{p}} = \nabla\Phi/p$ = unit normal to the wave front.

Here s is the arc length measured along the ray, c_0 is an arbitrary constant reference velocity, and Φ is the eikonal function (phase) of the wave front.

A. The Ordinary Vector Differential Equation

The time-independent eikonal equation for such a medium is well known[2-6], and can be derived as the short-wavelength limit (geometrical approximation) of the wave equation:

$$|\nabla\Phi|^2 = n^2(1 - \mathbf{V} \cdot \nabla\Phi/c_0)^2 \ . \qquad (1)$$

This can be written in terms of our terminology as

$$|\nabla\Phi| = p = \pm(n - p\mathbf{v} \cdot \hat{\mathbf{p}}) \ . \qquad (2)$$

We define now the direction of the acoustic ray path $\underset{\sim}{x}'$ to be the vector sum of the unit normal, $\hat{\underset{\sim}{p}} = \underset{\sim}{\nabla}\Phi/p$ and the velocity $\underset{\sim}{v}$ (see Fig. 1):

$$\hat{\underset{\sim}{p}} + \underset{\sim}{v} = q\underset{\sim}{x}' , \tag{3}$$

where q will be called the (nondimensional) <u>ray speed</u>. (Actually the true value of the ray speed is cq). This geometrical definition can be shown[5] to be identical with the physical definition that the ray path be the direction of energy transport. Multiplying Eq. 3 by p, we have

$$\underset{\sim}{\nabla}\Phi = N\underset{\sim}{x}' - \underset{\sim}{V} , \tag{4}$$

where we have introduced the two new auxiliary functions

$$N = pq \quad ; \quad \underset{\sim}{V} = p\underset{\sim}{v} . \tag{5}$$

The desired differential equation for the ray paths is obtained by taking the directional derivative of Eq. 4 in the direction of $\underset{\sim}{x}'$, and then eliminating Φ between these two equations[6]:

$$\frac{d}{ds}(N\underset{\sim}{x}' - \underset{\sim}{V}) = \frac{d}{ds}(\underset{\sim}{\nabla}\Phi) = x_i'\nabla_i\underset{\sim}{\nabla}\Phi = x_i'\underset{\sim}{\nabla}\nabla_i\Phi . \tag{6}$$

Note that we have interchanged the ∇_i and $\underset{\sim}{\nabla}$ operators in Eq. 6 and, therefore, $\Phi(\underset{\sim}{x})$ is subject to the usual continuity restrictions. We now substitute the i-th component of Eq. 4 into the last term of Eq. 6:

$$\frac{d}{ds}(N\underset{\sim}{x}') - \frac{d\underset{\sim}{V}}{ds} = x_i'\underset{\sim}{\nabla}(Nx_i' - V_i) = \underset{\sim}{\nabla}N - x_i'\underset{\sim}{\nabla}V_i . \tag{7}$$

In arriving at the last term of Eq. 7 we made use of the fact that $\underset{\sim}{r}'$ is a unit vector: $x_i' x_i' = 1$. Equation 7 is identical with an intermediary result of Ref. 1. However, at this point an error was made in Ref. 1 by taking $d\underset{\sim}{V}/ds = \underset{\sim}{r}' \cdot \nabla\underset{\sim}{V}$. $\underset{\sim}{V} = p(\underset{\sim}{r}, \underset{\sim}{r}')\underset{\sim}{v}(\underset{\sim}{r})$ is an explicit function of both the position $\underset{\sim}{r}$ and the ray direction $\underset{\sim}{r}'$ through that point, so that the correct expression for the directional derivative of $\underset{\sim}{V}$ must be

$$\frac{d\underset{\sim}{V}}{ds} = (\underset{\sim}{r}' \cdot \underset{\sim}{\nabla})\underset{\sim}{V} + (\underset{\sim}{r}'' \cdot \underset{\sim}{\nabla}')\underset{\sim}{V} \quad , \tag{8}$$

where $\underset{\sim}{\nabla}$ is the usual gradient operating on the position coordinates x_i only, while $\underset{\sim}{\nabla}' = \hat{\underset{\sim}{e}}_i \partial/\partial x_i'$ operates only on the direction coordinates x_i' .[7] Equation 7 now becomes

$$\frac{d}{ds}(N\underset{\sim}{r}') - (\underset{\sim}{r}'' \cdot \underset{\sim}{\nabla}')\underset{\sim}{V} + x_i' \underset{\sim}{\nabla} V_i - x_i' \nabla_i \underset{\sim}{V} = \underset{\sim}{\nabla} N \quad . \tag{9}$$

The last two terms in the left-hand side of Eq. 9 can be combined into a triple vector product, which yields as the final form of the differential equation for the rays

$$\frac{d}{ds}(N\underset{\sim}{r}') - (\underset{\sim}{r}'' \cdot \underset{\sim}{\nabla}')\underset{\sim}{V} + \underset{\sim}{r}' \times (\underset{\sim}{\nabla} \times \underset{\sim}{V}) = \underset{\sim}{\nabla} N. \tag{10}$$

Comparing this with Eq. 14 of Ref. 1 we see that the error there is the neglect of the term $(\underset{\sim}{r}'' \cdot \underset{\sim}{\nabla}')\underset{\sim}{V}$. Equation 10 is an ordinary second-order vector differential equation which can be integrated to give the ray path $\underset{\sim}{r}(s)$ as a function of its arc length s for any given initial values $\underset{\sim}{r}(0)$ and $\underset{\sim}{r}'(0)$.

B. The Functions N and $\mathbf{V}$

In order to integrate Eq. 10 we must express the functions N and $\mathbf{V}$ (defined in Eq. 5) explicitly in terms of $\mathbf{r}$ and $\mathbf{r}'$. Writing Eq. 3 as $\hat{\mathbf{p}} = q\mathbf{r}' - \mathbf{v}$ and squaring yields a second-degree algebraic equation for q which can be solved to give

$$q = \mathbf{r}'\cdot\mathbf{v} \pm S \quad ; \tag{11}$$

$$S = [1 - v^2 + (\mathbf{r}'\cdot\mathbf{v})^2]^{\frac{1}{2}} \quad . \tag{12}$$

Note from Fig. 1 that the geometrical interpretation of S is the projection of the unit vector $\hat{\mathbf{p}}$ on the ray direction $\mathbf{r}'$: $S = \cos(\alpha-\beta)$. In terms of the absolute velocities, therefore, cS is the projection of the wave velocity $c\hat{\mathbf{p}}$ in the direction of the ray. The requirement that $q > 0$ for $v < 1$ leads from the geometry of Fig. 1 to the conclusion that only the upper (+) sign in Eq. 11 is allowed. Similarly for $v \geq 1$ one can show that the upper sign must be used whenever

$$\cos\alpha = (\hat{\mathbf{p}}\cdot\mathbf{v})/v \geq -1/v \quad . \tag{13}$$

If the inequality in Eq. 13 is reversed then the lower (-) sign should be used in order to have $q \geq 0$. It is shown below, however, that the reversal of this inequality is strictly forbidden by another restriction on the function p, so that the upper sign in Eq. 11 holds under all conditions.[8] This additional restriction can easily be shown to be equivalent to the physical requirement that the rays be confined within the Mach cone

$$\cos\beta = (\mathbf{r}'\cdot\mathbf{v})/v \geq (v^2-1)^{\frac{1}{2}}/v \quad . \tag{14}$$

Equation 14 also guarantees the reality of the function S defined in Eq. 12.

To obtain $p(\underset{\sim}{r}, \underset{\sim}{r}')$ we solve Eq. 2:

$$p = n/(\underset{\sim}{v}\cdot\hat{\underset{\sim}{p}} \pm 1) \quad , \tag{15}$$

which proves the restriction in Eq. 13, because we must have $p > 0$. Substituting for $\underset{\sim}{v} \cdot \hat{\underset{\sim}{p}}$ from Eq. 3 (and using only the upper signs from now on) we can write Eq. 15 as

$$\begin{aligned} p &= n/(1 - v^2 + q\underset{\sim}{r}'\cdot\underset{\sim}{v}) = n/[1 - v^2 + (\underset{\sim}{r}'\cdot\underset{\sim}{v})^2 + (\underset{\sim}{r}'\cdot\underset{\sim}{v})S] \\ &= n/(\underset{\sim}{r}'\cdot\underset{\sim}{v} + S)S \quad . \end{aligned} \tag{16}$$

This can be written in a more convenient form after multiplying both numerator and denominator by $\underset{\sim}{r}'\cdot\underset{\sim}{v} - S$ and using Eq. 12:

$$p = n(S - \underset{\sim}{r}'\cdot\underset{\sim}{v})/(1 - v^2)S \quad . \tag{17}$$

The function N of Eq. 5 obtained by multiplication of Eqs. 11 and 17 now assumes a very simple form:

$$N = n/S \quad . \tag{18}$$

C. The Final Equations

The functions N and $\mathbf{V}$ are now expressed entirely in terms of $\mathbf{r}$ and $\mathbf{r}'$ by Eq's. 5, 11, 12, 17, and 18, so that the differential equation for the rays (Eq. 10) can be integrated for any given velocity $\mathbf{v}(\mathbf{r})$ and index of refraction $n(\mathbf{r})$. For easier reference all of the necessary equations are collected below:

$$\frac{d}{ds}(N\mathbf{r}') - (\mathbf{r}''\cdot\nabla')\mathbf{V} + \mathbf{r}' \times (\nabla \times \mathbf{V}) = \nabla N \quad ; \tag{19a}$$

$$N = n/S \quad ; \qquad \mathbf{V} = p\mathbf{v} \quad ; \tag{19b}$$

$$S = [1 - v^2 + (\mathbf{r}'\cdot\mathbf{v})^2]^{\frac{1}{2}} \quad ; \tag{19c}$$

$$p = n(S - \mathbf{r}'\cdot\mathbf{v})/(1 - v^2)S \quad . \tag{19d}$$

Note that for $v > 1$ and for a ray close to the Mach cone (approaching equality in Eq's. 13 and 14), S will approach zero, and therefore both N and $\mathbf{V}$ will grow beyond any bound. Also, for $v = 1$, independent of how close the ray is to the Mach cone, $S = \mathbf{r}'\cdot\mathbf{v}$, and therefore p and $\mathbf{V}$ become indeterminate. In both of these cases, therefore, one should expect difficulties when integrating Eq. 19a numerically.

II. THE CHARACTERISTICS OF THE EIKONAL EQUATION

Courant and Hilbert[9] have shown that in the trivial case $\mathbf{v}=0$, $n=1$ the sound rays are the characteristics of the eikonal equation, Eq. 1. In this section we shall prove this to be the case also for our general problem. Besides showing that Eq. 19a is the differential equation for the characteristics this method will also give us a

somewhat simpler equation for the rays.

The theory of partial differential equations[1] tells us that any first-order differential equation for Φ

$$H(x_i, p_i, \Phi) = 0 \quad ; \quad p_i \equiv \partial\Phi/\partial x_i \quad , \tag{20}$$

is solved by a family of characteristics which are curves given by the following system of ordinary differential equations:

$$dx_i/d\tau = \partial H/\partial p_i \quad , \tag{21a}$$

$$dp_i/d\tau = -(\partial H/\partial x_i + p_i \partial H/\partial\Phi) \quad , \tag{21b}$$

$$d\Phi/d\tau = p_j \partial H/\partial p_j \quad , \tag{21c}$$

where τ is an independent parameter. The eikonal equation (Eq. 1) written in the form of Eq. 20 becomes

$$H = p_j p_j - (n - v_j p_j)^2 = 0 \quad , \tag{22}$$

and its characteristics as given by Eq's. 21a and 21b are easily shown to be:

$$\dot{x}_i = 2[\,p_i + (n - p_j v_j) v_i\,] \quad , \tag{23a}$$

$$\dot{p}_i = 2(n - v_j p_j)[(\partial n/\partial x_i) - p_j(\partial v_j/\partial x_i)] \quad , \tag{23b}$$

where from now on a dot will signify differentiation with respect to τ. Equation 21c gives the value of the eikonal function Φ on the characteristics which, however, is not needed for tracing the rays. The significance of Eq. 21c will be explored further in the Appendix. Equation 23a is, apart from a scale factor, equivalent to Keller's[2] Eq. 38. Keller, however, does not consider the second set of the

characteristic equations, Eq. 23b, which are needed to eliminate the p_i from Eq. 23a in order to obtain a differential equation for the rays. Equations 23 with the help of Eq. 22 can be written more conveniently in vector form[11]:

$$\dot{\mathbf{x}} = 2(\mathbf{p} + p\mathbf{v}) \tag{24a}$$

$$\dot{\mathbf{p}} = 2p(\nabla n - p_j \nabla v_j) \quad . \tag{24b}$$

Comparing Eq. 24a with Eq. 3 we see indeed that the direction $\dot{\mathbf{x}}$ of the characteristics coincides with the direction $\mathbf{x}'$ of the sound rays. It still remains to be shown, however, that Eq's. 24 are identical to Eq. 10. To do this we first eliminate $\mathbf{p}$ between Eq's. 24 by solving for $\mathbf{p}$ from Eq. 24a:

$$\mathbf{p} = \frac{1}{2}\dot{\mathbf{x}} - p\mathbf{v} \quad . \tag{25}$$

Taking the dot product of this equation with $\mathbf{v}$ and using

$$\mathbf{p} \cdot \mathbf{v} = n - p \tag{26}$$

from Eq. 2, allows us also to determine explicitly p (the magnitude of the vector $\mathbf{p}$):

$$p = (n - \tfrac{1}{2}\dot{\mathbf{x}}\cdot\mathbf{v})/(1 - v^2) \quad . \tag{27}$$

Substituting Eq. 25 into Eq. 24b we get

$$\frac{d}{d\tau}\left(\frac{1}{2}\dot{\mathbf{x}} - p\mathbf{v}\right) = 2p\left(\nabla n - p_j \nabla v_j\right) \quad . \tag{28}$$

Equation 28 (together with Eq's. 25 and 27 which are needed for p and p_j in Eq. 28) is a single second-order vector differential equation

for the ray paths. It looks somewhat simpler than Eq. 19 and therefore might be preferred for practical calculations. The integration of Eq. 28, however, will give the rays in terms of the as yet undefined canonical parameter τ, whereas Eq. 19 gives them as functions of the physical arc length s. Whether one is to be preferred over the other is not known in general, and might depend on the particular problem under consideration. For example,[1,12] in some situations a constant integration step Δs is desirable, in which case the formulation of Eqs. 19 would be preferred. In other cases[13], however, where the ray might wind itself into a point of singularity, a constant Δs would be completely inadequate for numerical integrations.

To complete the identification of Eq. 28 with Eq. 19, and at the same time to determine the physical meaning of the canonical variable τ, we make a transformation of the independent variable from τ to s in Eq. 28. This is done easiest by substituting

$$\dot{\underset{\sim}{r}} = \underset{\sim}{r}'(ds/d\tau) \tag{29}$$

into Eq. 25, solving for $\underset{\sim}{r}'$, and then squaring. The result can be simplified by noting that $|\underset{\sim}{r}'|^2 = 1$, and by using Eqs. 26 and 27 to substitute for all factors containing p or $\underset{\sim}{p}\cdot\underset{\sim}{r}$. A little algebra will finally yield

$$ds/d\tau = 2n/S \quad , \tag{30}$$

where S is the same function of $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ which was used in section I and is defined in Eq. 19c. Equations 29 and 30 can now be used to

rewrite Eqs. 28 and 27 with s as the independent variable:

$$\frac{d}{ds}\left(\frac{n\underset{\sim}{x}'}{S} - p\underset{\sim}{v}\right) = \frac{nS}{n}\left(\underset{\sim}{\nabla}n - p_j\underset{\sim}{\nabla}v_j\right) \quad , \tag{31a}$$

$$p = n(S - \underset{\sim}{x}'\cdot\underset{\sim}{v})/(1 - v^2)S \quad . \tag{31b}$$

Note that Eq. 31b agrees exactly with Eq. 19d, and that the left side of Eq. 31a agrees with the left side of Eq. 7. It is straightforward but rather tedious to show that the right sides of Eqs. 31a and 7 agree also. Therefore we have proven our contention that the characteristics of the eikonal equation are the acoustic rays, and as a by-product have obtained an alternative (possibly simpler) formulation, Eqs. 28, 25 and 27, for the ray differential equation.

It is interesting to note that part of the troubles anticipated with a numerical integration of Eq. 19a do not appear in the new formulation of Eq. 28. That is, for $v > 1$, there are no quantities which blow up when a ray approaches the Mach cone. For $v=1$, however, the same difficulty persists as in Eqs. 19, because p (Eq. 27) then becomes infinite. We can understand better now why the first difficulty appears in Eqs. 19 and is removed in the formulation of Eq. 28: Eq. 30 shows that close to the Mach cone, where S approaches zero, $ds/d\tau$ approaches infinity, so that an infinitesimal integration step $\Delta\tau$ in terms of the canonical variable corresponds to an infinite step in terms of the arc length.

III. THE LAGRANGIAN AND FERMAT'S PRINCIPLE

As was mentioned in the Introduction, the method of characteristics which was used to derive the ray differential equation in section II is equivalent to the Hamiltonian method used by Keller[2]. In fact, since the function H of Eq. 22 does not depend explicitly on Φ, the differential equations for the characteristics, Eqs. 21, are precisely Hamilton's canonical equations of motion which are well known in classical mechanics[14]. The function $H(x_i, p_i)$, therefore, can be regarded as the Hamiltonian of some mechanical system, and with it we can construct the Lagrangian from which the ray differential equation is derivable via a variational principle (Lagrange's equations).

The Lagrangian $L(x_i, \dot{x}_i)$ is given in terms of the Hamiltonian $H(x_i, p_i)$ by[14]

$$\begin{aligned} L &= \dot{x}_i p_i - H \\ &= \dot{x}_i p_i - p_i p_i + (n - v_i p_i)^2 \quad . \end{aligned} \tag{32}$$

In order to put the Lagrangian in its proper form (a function of the x_i and $\dot{x}_i$ only) we must solve the system of Eqs. 23a for the p_i in terms of the x_i (without using the "energy" equation, Eq. 22), and substitute the result into Eq. 32. The algebra is rather lengthy, so we shall omit it here and just give the final result:

$$L = \frac{\left(n - \frac{1}{2}\dot{\underline{x}}\cdot\underline{v}\right)^2}{1 - v^2} + \frac{1}{4}|\dot{\underline{x}}|^2 \quad . \tag{33}$$

Lagrange's equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \tag{34}$$

can be used to derive the ray differential equation (Eq. 28) once more, proving that Eq. 33 is indeed the correct Lagrangian for this problem.

Having obtained the Lagrangian we can now examine the variational principle

$$\delta \int_A^B L \, d\tau = 0 \quad , \tag{35}$$

(whose solution is Eq. 34) which for the case of a motionless medium gives the well-known Fermat's principle, which states that the ray path between any two points A,B is that curve along which the time of travel is minimum. Kornhauser[8] implied that such a Fermat's principle may not exist for a moving medium. He also asserted that Rayleigh's[15] contention, that the velocity to be used in Fermat's principle is the speed of sound plus the component of the fluid velocity in the ray direction, is incorrect. To see whether Eq. 35 is a Fermat's principle (in the sense of minimizing a travel time) we rewrite the Lagrangian in terms of the arc length variable s (using Eqs. 29, 30 and 31b)

$$L = 2np \quad , \tag{36}$$

whereupon Eq. (35) can be written

$$\delta \int_A^B L\, d\tau = \delta \int_A^B 2np\left(\frac{d\tau}{ds}\right)ds = \delta \int_A^B pS ds$$

$$= \delta \int_A^B \frac{n}{S + \underset{\sim}{r}' \cdot \underset{\sim}{v}}\, ds = 0 \quad , \qquad (37)$$

where in the last step we have used Eq. (16) to substitute for p. The denominator of this last expression is just the <u>ray speed</u> q defined in Eq. 11 so that the variational integral

$$\delta \int_A^B \frac{ds}{cq} = 0 \qquad (38)$$

is indeed Fermat's principle in the strict sense, because it minimizes the ray's travel time. Referring to Fig. 1 and using Eq. 3 we see that the velocity to be used in the denominator of Eq. 38 is $cq = c \cos(\alpha - \beta) + \tilde{v} \cos \beta$, whereas Rayleigh's[16] idea was to use $c + \tilde{v} \cos \beta$. Kornhauser[3], therefore, was correct in saying that Fermat's principle with the velocity proposed by Rayleigh is wrong. We see however that the rays do obey Fermat's principle, the correct expression of which is Eq. 38.

IV. THE RAY EQUATIONS FOR PLANAR RAYS IN A MEDIUM WITH $v^2 \ll 1$.

If the speed of the medium $\tilde{v}$ at every point is much smaller than the local speed of sound c (which is the case for most propagation problems in either the ocean or the atmosphere) the differential equations for the rays (Eqs. 19 or 28) admit considerable simplifications. Keeping only first-order terms in v (as compared to unity)

the functions N and $\underset{\sim}{V}$ of Eq. 19 reduce to

$$N = n; \qquad \underset{\sim}{V} = n(1 - \underset{\sim}{r}'\cdot\underset{\sim}{v})\underset{\sim}{v} \quad . \tag{39}$$

We shall consider a further simplification by assuming that n and $\underset{\sim}{v}$ depend only on x and y, and that $\underset{\sim}{v}$ lies in the x-y plane:

$$n = n(x,y) \quad ; \qquad \underset{\sim}{v} = v_x(x,y)\underset{\sim}{i} + v_y(x,y)\underset{\sim}{j} \quad ; \tag{40}$$

where $\underset{\sim}{i}$, $\underset{\sim}{j}$ are the usual Cartesian unit vectors in the x,y directions. In that case rays which start out in the x-y plane will always remain in that plane. Such rays, therefore, can be described by their curvature κ, given by

$$\underset{\sim}{r}'' = \kappa\hat{\underset{\sim}{\beta}} \quad , \tag{41}$$

where $\hat{\underset{\sim}{\beta}}$ is the unit normal vector,

$$\hat{\underset{\sim}{\beta}} = -\, y'\underset{\sim}{i} + x'\underset{\sim}{j} \quad , \tag{42}$$

which is perpendicular to the unit tangent vector

$$\underset{\sim}{r}' = x'\underset{\sim}{i} + y'\underset{\sim}{j} \quad . \tag{43}$$

The curvature κ, as defined in Eq. (41) can be obtained directly from Eq. 19a. The term $(\underset{\sim}{r}''\cdot\underset{\sim}{V}')\underset{\sim}{V}$ in that equation is of the order $nv^2\underset{\sim}{r}''$, and therefore, compared with the first term $n\underset{\sim}{r}''$, is negligible. The third term can be worked out to give

$$\underset{\sim}{r}' \times (\underset{\sim}{\nabla} \times \underset{\sim}{V}) = (\partial v_x/\partial y - \partial v_y/\partial x)\hat{\underset{\sim}{\beta}} \quad , \tag{44}$$

with

$$V_i = nv_i(1 - \underset{\sim}{r}'\cdot\underset{\sim}{v}) \quad ; \qquad i = x,y \quad . \tag{45}$$

The last term ($\underset{\sim}{\nabla}n$), and part of the first term ($\underset{\sim}{r}'dn/ds$) can be combined by using the identities $dn/ds = \underset{\sim}{r}'\cdot\underset{\sim}{\nabla}n$ and $x'^2 + y'^2 = 1$ to yield

$$\underset{\sim}{\nabla}n - (dn/ds)\underset{\sim}{r}' = (x'\partial n/\partial y - y'\partial n/\partial x)\underset{\sim}{\beta} \quad . \tag{46}$$

Equation 19a then becomes

$$n\underset{\sim}{r}'' = (x'\partial n/\partial y - y'\partial n/\partial x + \partial V_y/\partial x - \partial V_x/\partial y)\underset{\sim}{\beta} \quad , \tag{47}$$

whereupon comparison with Eq. 41 gives for the curvature

$$\kappa = n^{-1}(x'\partial n/\partial y - y'\partial n/\partial x + \partial V_y/\partial x - \partial V_x/\partial y) \quad . \tag{48}$$

For $\underset{\sim}{v} = 0$ this equation agrees with the result first derived in Ref. 16. Equation 48 is still quite complicated. It is a second-order differential equation for x(s), y(s), which together with $x'^2 + y'^2 = 1$ can be integrated to obtain the ray paths in parametric form with the arc length s as the parameter. Or one can eliminate the parameter s, and obtain the equation of the ray path in the form y = y(x) directly, as the solution of a single second-order equation for y. In either case, however, the solutions usually would have to be obtained by a numerical integration.

If the medium is simplified somewhat more (as in Ref. 1), analytical solutions of Eq. 48 are possible, at least in terms of a quadrature.

If the index of refraction n and the velocity of the medium $\underset{\sim}{v}$ are functions of only one coordinate, say y, and furthermore, if $\underset{\sim}{v} = v(y)\underset{\sim}{i}$ has only a component in the x-direction, then Eq. 48 simplifies further:

$$\kappa = n^{-1}(x'dn/dy - \partial V/\partial y) \quad , \tag{49}$$

where

$$V = nv(1 - vx') \quad . \tag{50}$$

Substituting Eq. 50 into Eq. 49, and using the relation $(dn/dy)/n = -(dc/dy)/c$ one can obtain the following expression for the curvature:

$$\kappa = c^{-1}(dc/dy)(v-x') + (dv/dy)(2x'v-1) \quad , \tag{51}$$

which is identical[17] to Eq. 24 of Ref. 1. Thus, even though the general ray differential equation in Ref. 1 is incorrect, the small-velocity approximation is correct. In particular, the analytic solution in terms of a quadrature for a medium with a constant wind $(d(vc)/dy = 0)$, as well as the two applications (atmosphere with a constant speed-of-sound gradient and atmosphere with a constant temperature gradient) are correct. The reason that the sections of Ref. 1 which deal with the small-velocity approximation are correct is, of course the fact, that the term $(\underset{\sim}{r}''\cdot\underset{\sim}{\nabla}')\underset{\sim}{V}$ in Eq. 19a, whose omission is the error of Ref. 1, is negligible in this approximation.

APPENDIX A

In this appendix a brief account is given of the theory of the eikonal equation and of the geometry of the rays and wave normals for subsonic as well as supersonic flows.

The wave surfaces are described by the equation

$$\Phi = c_0 t \quad , \tag{A1}$$

where Φ is a solution of the eikonal equation, Eq. 1, and t is the time from an arbitrary origin. From this it easily follows that the unit vector $\hat{\underset{\sim}{p}}$ points in the direction in which the wave moves and that p is the reciprocal wave speed multiplied by c_0. Let us introduce the following notation for the non-dimensional wave speed

$$w = \frac{c_n}{cp} = \frac{n}{p} \quad . \tag{A2}$$

It is a quantity which by definition is nonnegative. The eikonal equation implies that

$$w = \underset{\sim}{v}\cdot\hat{\underset{\sim}{p}} \pm 1 \tag{A3}$$

which is equivalent to Eq. 2. This equation gives the speed at which a wavelet moves provided its normal and direction of motion are known. One could use it in principle to calculate the motion of a wavefront if it were not for the ambiguity in the sign of the second term in the right-hand member and the sense of the unit vector $\hat{\underset{\sim}{p}}$.

To clear up these ambiguities let us investigate the Cauchy problem of continuing a wave given at an instant of time, say $t = 0$. It is convenient to introduce a local right-handed Cartesian coordinate system at the wave. Let n (not to be confused with the index of refraction) be the coordinate normal to the surface, so that

$$v_n = \underset{\sim}{v} \cdot \hat{\underset{\sim}{n}} \geq 0$$

where $\hat{\underset{\sim}{n}}$ is the unit vector normal to the wave and pointing into the half space into which $\underset{\sim}{v}$ points. The eikonal equation becomes

$$\frac{1}{n^2} \Phi_n^2 = \left(1 - \frac{1}{n} v_n \Phi_n\right)^2 \quad . \tag{A4}$$

This quadratic equation for Φ_n has the solutions

$$\frac{1}{n} \Phi_n = \frac{1}{v_n \pm 1} \quad . \tag{A5}$$

Therefore, provided $v_n - 1 \neq 0$ on the initial surface, there are two possible solutions to the Cauchy problem. One may introduce the concept of "advancing" and "receding" wavelets with respect to the flow. The plus sign refers to the advancing wavelet and the minus sign to the receding wavelet. Equations A5, A2 and the definition of p imply that the wave speed is given by

$$w = \frac{n}{|\Phi_n|} = |v_n \pm 1| \quad . \tag{A6}$$

This implies that the advancing wavelet has a greater wave speed than the receding wavelet. When $v_n = 0$ there is no distinction between

advancing and receding wavelets and the local motion of the wavelet is like that in a medium at rest. Leaving aside for the moment the possibility that $v_a - 1 = 0$ on the initial wave surface, one may now determine the sign to be taken in Eq. A3 or equivalently Eq. 2. The results up to this point are given in Table A1 as the first four entries for each of the four possible cases (advancing or receding wavelets in subsonic or supersonic flow). It is clear that $v_a > 1$ can occur only in a supersonic flow. When $\mathbf{v}\cdot\hat{\mathbf{p}} > 0$, the fluid velocity vector points into the half space into which the wavelet is moving, when $\mathbf{v}\cdot\hat{\mathbf{p}} < 0$ the converse holds.

We have now determined what the sign should be in the eikonal equation, Eq. 2, and the equivalent Eq. A3, which gives the wave speed as a function of the wave normal $\hat{\mathbf{p}}$. We have also determined what the sense of the wave normal $\hat{\mathbf{p}}$ is in relation to the velocity $\mathbf{v}$.

Instead of advancing the wave normal to itself using Eq. A3, it is more convenient to advance each wavelet along the ray using the system of ordinary differential equations 23a, 23b. We wish to study these equations in greater detail in order to remove the ambiguity in the sign of Eqs. 24a and 24b (see Refs. 11 and 8), and also to find a simple geometric construction relating rays, wave normals, ray speeds and wave speeds.

To the characteristic equations 23a, 23b one may, using Eq. 21c, add the following relation:

$$\frac{d\Phi}{d\tau} = 2n(n - v_i p_i) = \pm 2n(p_i p_i)^{\frac{1}{2}} = \pm 2np \quad , \tag{A7}$$

where use has been made of the eikonal equation. This equation combined with Eq. A1 gives

$$c_0 \frac{dt}{d\tau} = \frac{d\Phi}{d\tau} = \pm 2np \quad , \tag{A8}$$

or

$$c \frac{dt}{d\tau} = \pm 2 \frac{n}{w} \tag{A9}$$

which, apart from the ambiguity in the sign, relates the parameter τ to physically significant quantities. The rule of signs to be followed is the same as in the eikonal equation written in the form of Eq. A3.

One may now cast the characteristic equations 23a in a geometrically perspicuous form. Introducing time as independent variable

instead of τ one gets from Eq. 23a

$$\frac{1}{c_0}\frac{dx_i}{dt} = \frac{1}{c_0}\,\dot{x}_i\,\frac{d\tau}{dt} = \frac{1}{n}\left\{\pm\frac{p_i}{p} + \quad v_i\right\} \tag{A10}$$

or

$$\frac{1}{c}\frac{dx_i}{dt} = v_i \pm \frac{p_i}{p} \ . \tag{A11}$$

In the notation of this paper this may be written

$$q\underset{\sim}{r}' = \underset{\sim}{v} \pm \hat{\underset{\sim}{p}} \ . \tag{A12}$$

This is the generalization of Eq. 3 and applies also to supersonic flow. The results obtained are collected in Table A1. This permits us to give the relationship between ray, wave normal, ray speed and wave speed in graphical form (see Figures A1 and A2 for the subsonic and supersonic cases respectively). In either case the circle is of unit radius centered at 0 and the line $\overrightarrow{AO}$ represents the velocity vector $\underset{\sim}{v}$.

In the subsonic case, ($v < 0$), which is normal in acoustics, the situation is quite simple as indicated in Fig. A1. The vector $\overrightarrow{AB}$ represents the ray. Its length is q, the dimensionless ray speed. The ray is made up of the vector sum of the velocity $\underset{\sim}{v}$ and the unit wave normal $\hat{\underset{\sim}{p}}$ represented by $\overrightarrow{OB}$ (see Table A1). The projection of $\underset{\sim}{v}$ on $\hat{\underset{\sim}{p}}$ plus unity is the wave speed represented by the length of CB. The right half of the circumference of the circle represents advancing wavelets and the left half receding wavelets.

The relationships for the supersonic case ($v > 1$) are more involved and are indicated in Fig. A2. A line drawn from A to a point

on the circle represents a ray as before. One has, for example, $\overrightarrow{AB}$ and $\overrightarrow{AB'}$ two rays with the same direction, one fast and one slow. The unit normal $\hat{p}$ pertaining to $\overrightarrow{AB}$ is $\overrightarrow{OB}$ and that pertaining to $\overrightarrow{AB'}$ is $\overrightarrow{B'O}$. The wave velocity vector $w\hat{p}$ pertaining to the two rays are $\overrightarrow{CB}$ and $\overrightarrow{C'B'}$ respectively. We see that by considering only the plus sign in Eq. 3 we confine ourselves to the fast rays in the supersonic case. In the supersonic case we note further that the angle β between the ray and the velocity vector must satisfy

$$|\beta| \le \mu$$

where

$$\sin \mu = \frac{1}{v} \tag{A13}$$

and μ is called the Mach angle. Thus one may not start a ray with an arbitrary slope.

From the graphic representation one may derive various relationships by trigonometry. We give some below.

The non-dimensional ray speed as a function of the angle the ray makes with the velocity vector is

$$q = v \cos\beta \pm v \left(\frac{1}{v^2} - \sin^2\beta\right)^{1/2}. \tag{A14}$$

When $v < 1$ (subsonic flow) the plus sign is to be used. When $v > 1$ (supersonic flow) the two signs give the fast and slow rays respectively.

The cosine of the angle between ray and wave normal (it is always an acute angle) is given by

$$\underline{r}'\cdot\hat{\underline{p}} = v\left(\frac{1}{v^2} - \sin^2\beta\right)^{1/2}, \tag{A15}$$

which is identical to the quantity S defined before. The dimensionless wave speed may be found from

$$w = q\underline{r}'\cdot\hat{\underline{p}} \quad . \tag{A16}$$

In conclusion one may now see what the meaning of $v_a - 1 = 0$ is, a case that was excluded from the discussion. It pertains to a receding wavelet carried along by the fluid so that its normal speed vanishes. It corresponds to the ray $\vec{AB''}$ in Fig. A2. Clearly on such a wavelet $|\mathrm{grad}\ \Phi| = \frac{\Omega}{w} = \infty$ and the ray $\vec{AB''}$ is tangent to the wavelet. At such a wavelet Eq. A1 cannot hold. In mathematical language the eikonal function Φ must be singular there. The surfaces on which $w = 0$ appear as branch surfaces of Φ, joining the fast and slow waves.

REFERENCES

1. P. Uginčius, J. Acoust. Soc. Amer. **37**, 476-479 (1965).
2. J. B. Keller, J. Appl. Phys. **25**, 938-947 (1954).
3. E. T. Kornhauser, J. Acoust. Soc. Amer. **25**, 945-949 (1953).
4. P. M. Morse and K. U. Ingard, Encyclopedia of Physics, S. Flügge, Ed. (Springer Verlag, Berlin, 1961), Vol. 11/1, Acoustics I, p. 80, Eq. 32.3.
5. D. I. Blokhintsev, Acoustics of a Nonhomogeneous Moving Medium, Natl. Adv. Comm. Aeronaut. Tech. Mem. 1399 (1956), Eq. 2.14.
6. The summation convention will be used throughout. Cartesian unit vectors will be denoted by $\hat{\mathbf{e}}_i$.
7. Note that $d\Phi/ds = (\mathbf{r}'\cdot\nabla)\Phi$ which was used to obtain Eq. 6 is correct, since Φ being a solution of Eq. 1 is a function of $\mathbf{r}$ only.
8. This is not quite true. For supersonic flows the eikonal equation has two different solutions $\hat{\mathbf{p}}$ with different ray speeds q, both giving the same ray direction $\mathbf{r}'$ as defined in Eq. 3. In choosing only the upper sign in Eq. 11 we limit ourselves to the solution with the faster ray speed. This is explained in more detail in the Appendix.
9. R. Courant and D. Hilbert, Methods of Mathematical Physics (Interscience Publishers, Inc., New York, 1962), Vol. 2.
10. See Ref. 9 or any other text on partial differential equations.
11. When taking the square root of Eq. 22 we again get a (±) sign: $n - v_j p_j = \pm p$. Exactly the same arguments as in section I apply, however, and the lower sign, therefore, is extraneous.

12. H. J. Lugt and P. Uginčius, J. Acoust. Soc. Amer. 36, 689-694 (1964).

13. R. F. Salant, J. Acoust. Soc. Amer. 46, 1153-1157 (1969).

14. H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, Mass., 1959) ch. 7.

15. Lord Rayleigh, Theory of Sound (Dover Publications, New York, 1945), Vol. II, p. 132.

16. P. Uginčius, J. Acoust. Soc. Amer. 45, 193-205 (1969), Eq. 19.

17. Note that v in Ref. 1 is the velocity of the medium, whereas here the velocity is vc. When this is taken into account Eq. 51 agrees exactly with Eq. 24 of Ref. 1.

	Advancing Wavelet		Receding Wavelet	
Subsonic Normal Speed $v_n < 1$	$w = v_n + 1$	A6	$w = -v_n + 1$	A6
	$v_n = \mathbf{v}\cdot\hat{\mathbf{p}} > 0$		$v_n = -\mathbf{v}\cdot\hat{\mathbf{p}} > 0$	
	$w = \mathbf{v}\cdot\hat{\mathbf{p}} + 1$	A3	$w = \mathbf{v}\cdot\hat{\mathbf{p}} + 1$	A3
	$p = n - p\mathbf{v}\cdot\hat{\mathbf{p}}$	2	$p = n - p\mathbf{v}\cdot\hat{\mathbf{p}}$	2
	$n - v_t p_t = (p_t p_t)^{\frac{1}{2}}$		$n - v_t p_t = (p_t p_t)^{\frac{1}{2}}$	
	$cdt = 2\,\frac{n}{w}\,d\tau$	A9	$cdt = 2\,\frac{n}{w}\,d\tau$	A9
	$\frac{dx_t}{cdt} = v_t + \frac{p_t}{p}$	A11	$\frac{dx_t}{cdt} = v_t + \frac{p_t}{p}$	A11
	$q\mathbf{r}' = \mathbf{v} + \hat{\mathbf{p}}$	A12	$q\mathbf{r}' = \mathbf{v} + \hat{\mathbf{p}}$	A12
Supersonic Normal Speed $v_n > 1$	$w = v_n + 1$	A6	$w = v_n - 1$	A6
	$v_n = \mathbf{v}\cdot\hat{\mathbf{p}} > 0$		$v_n = \mathbf{v}\cdot\hat{\mathbf{p}} > 0$	
	$w = \mathbf{v}\cdot\hat{\mathbf{p}} + 1$	A3	$w = \mathbf{v}\cdot\hat{\mathbf{p}} - 1$	A3
	$p = n - p\mathbf{v}\cdot\hat{\mathbf{p}}$	2	$p = -(n - p\mathbf{v}\cdot\hat{\mathbf{p}})$	2
	$n - v_t p_t = (p_t p_t)^{\frac{1}{2}}$		$n - v_t p_t = -(p_t p_t)^{\frac{1}{2}}$	
	$cdt = 2\,\frac{n}{w}\,d\tau$	A9	$cdt = -\,2\,\frac{n}{w}\,d\tau$	A9
	$\frac{dx_t}{cdt} = v_t + \frac{p_t}{p}$	A11	$\frac{dx_t}{cdt} = v_t - \frac{p_t}{p}$	A11
	$q\mathbf{r}' = \mathbf{v} + \hat{\mathbf{p}}$	A12	$q\mathbf{r}' = \mathbf{v} - \hat{\mathbf{p}}$	A12

TABLE A1. The basic equations for advancing and receding wavelets in subsonic and supersonic flows.

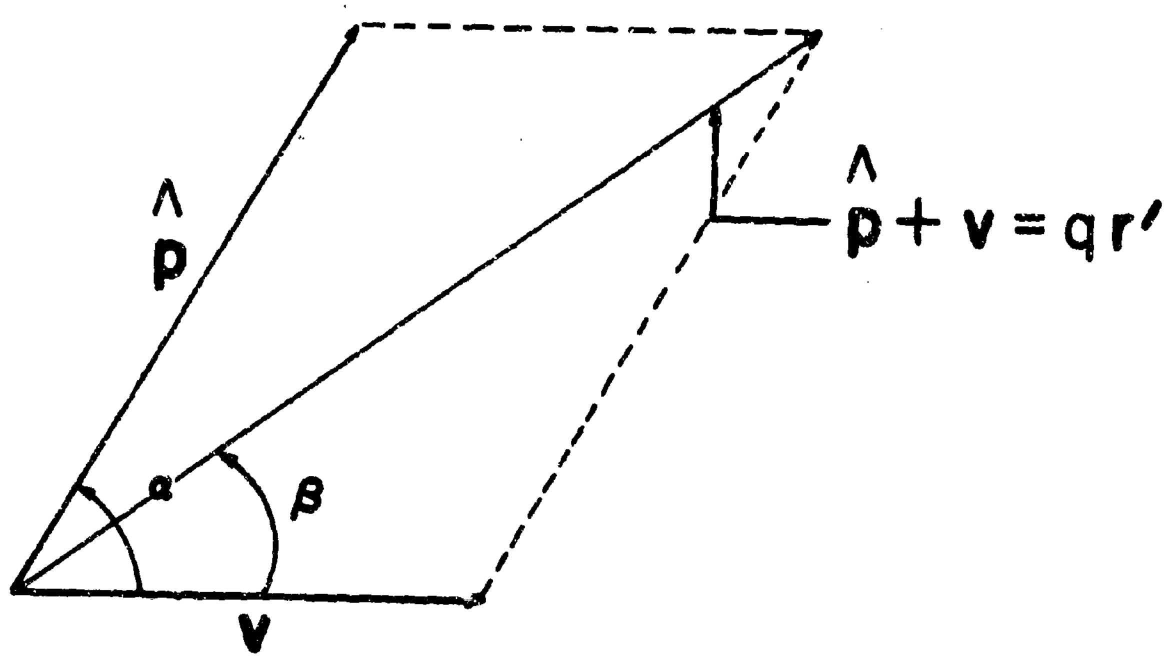

Fig. 1: Geometrical definition of the ray direction r'.

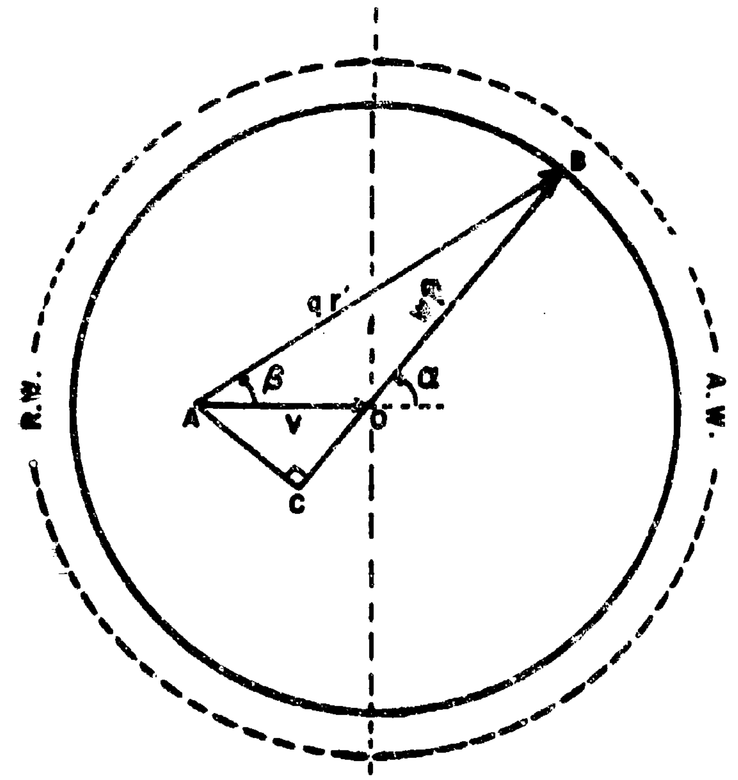

Fig. A-1: Relationship between rays and wave normals at a subsonic point. A. W.: advancing wavelet. R. W.: receding wavelet.

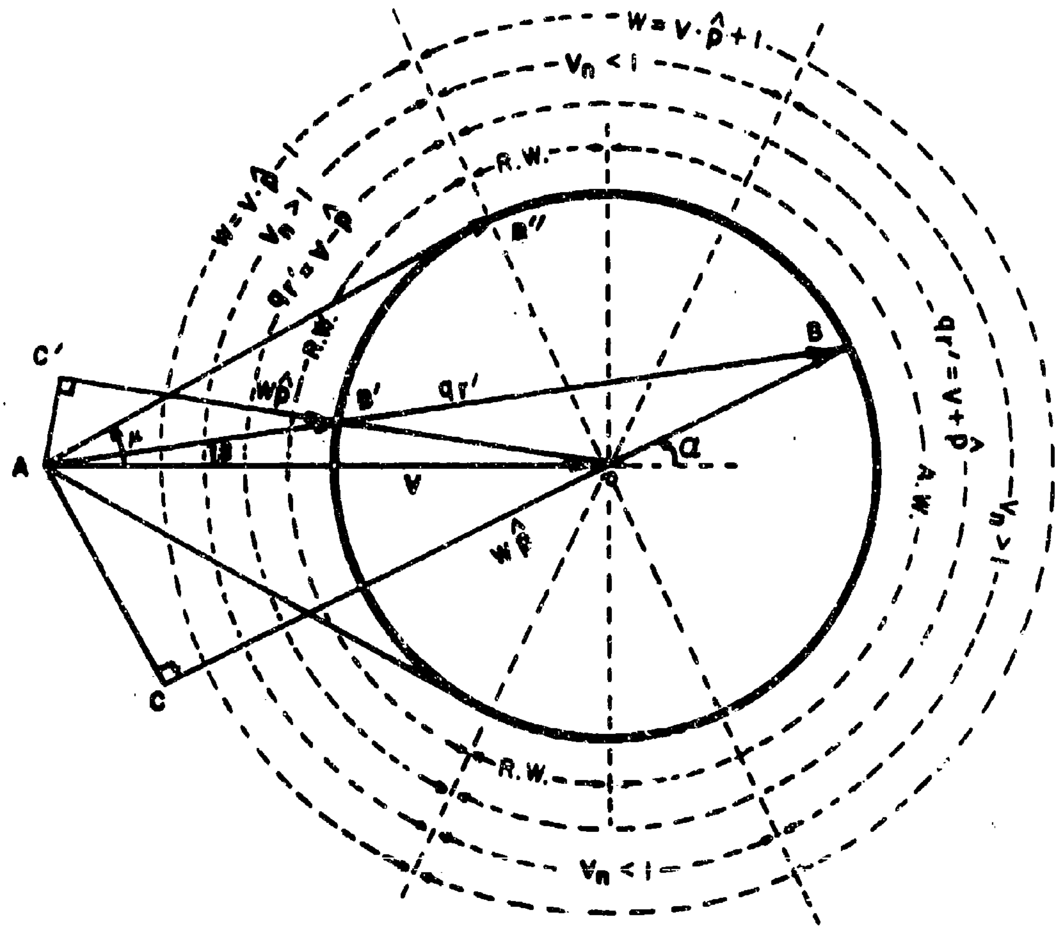

Fig. A-2: Relationship between rays and wave normals at a supersonic point. A. W.: advancing wavelet, R. W.: receding wavelet.

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) U. S. Naval Weapons Laboratory Dahlgren, Virginia	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

ACOUSTIC RAYS IN AN ARBITRARY MOVING INHOMOGENEOUS MEDIUM

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

Peter Uginicius
Bernd Zondek

6. REPORT DATE	7a. TOTAL NO. OF PAGES 34	7b. NO. OF REFS

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S) TR-2446
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

Two different formulations are presented in terms of ordinary second-order vector differential equations for acoustic rays in a three-dimensional medium moving with an arbitrary velocity (either subsonic or supersonic), and having an arbitrary index of refraction. The first formulation has the arc length of the ray as independent variable, while the second one is given in terms of a canonical varible and is equivalent to Keller's Hamiltonian formulation. (J. B. Keller, J. Appl. Phys. 25, 938-947 (1954)).

DD FORM 1473 1 NOV 65 (PAGE 1)
S/N 0101-807-6811

UNCLASSIFIED
Security Classification

A-31408